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Functional analysis 2
Spring 2025
Sheet 12

Exercises for 'Functional Analysis 2' [MATH-404]

(19/05/2025)

Ex 12.1 (An algorithm for Edelstein's fixed point theorem)
We saw in the lecture that when (M, d) is a compact metric space and $F : M \to M$ satisfies $d(F(x), F(y)) < d(x, y)$ for all $x \neq y$, then F has a unique fixed point $\bar{x} \in M$. Show that for any $x_0 \in M$ the iteratively defined sequence $x_{n+1} = F(x_n)$ converges to $\bar{x}$.

Ex 12.2 (Schaefer's fixed point theorem)
Let X be a Banach space and $F : X \to X$ be continuous such that $\overline{F(B)}$ is compact for every bounded set $B \subset X$. Assume further that there exists $R > 0$ such that

$$\{x \in X : x = \lambda F(x) \text{ for some } \lambda \in [0, 1]\} \subset B_R(0).$$

Show that F has a fixed point.
Hint: Define the projection operator $p_R : X \to \overline{B_R(0)}$ by $p_R(x) = x$ on $\overline{B_R(0)}$ and $p_R(x) = R\frac{x}{|x|}$ otherwise and consider the map $F_R = p_R \circ F$. Apply Schauder's fixed point theorem on a suitable set.

Ex 12.3 (Peano's existence theorem for ODEs*)
Let $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ and consider the Cauchy problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \tag{1}$$

Assume that $f : [-a + t_0, a + t_0] \times \overline{B_R(y_0)} \to \mathbb{R}^n$ is continuous. Show that there exists $\delta > 0$ such that the Cauchy problem (1) has a solution $y : [-\delta + t_0, \delta + t_0] \to \mathbb{R}^n$.
Hint: Apply the Schauder fixed point theorem to the integral operator $y \mapsto y_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s$. Use the Arzelà–Ascoli theorem to show the compactness of the operator.

Ex 12.4 (Existence of solutions for a periodic BVP)
Let $\mu \in \mathbb{R} \setminus \{0\}$ and $\mathbb{J} = [0, T]$ for some $T > 0$.

a) Consider the linear first order periodic boundary value problem

$$u'(t) + \mu u(t) = f(t), \quad t \in J, \quad u(0) = u(T),$$

where $f \in C(\mathbb{J})$. Find the Green's function $g(t, s)$ such that

$$u(t) = [Gf](t) := \int_0^T g(t, s) f(s)\,\mathrm{d}s, \quad t \in \mathbb{J},$$

is a solution to this problem.
Hint: Consider the function $y(t) = e^{\mu t} u(t)$.

b) Show that $G\colon C(\mathbb{J}) \to C(\mathbb{J})$ is continuous and maps bounded subsets of $C(\mathbb{J})$ into relatively compact sets.

c) Assume that $f\colon \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is continuous and has *sublinear growth*, i.e.

$$|f(t,u)| \le a(t) + b|u|^\alpha,$$

where $a \in C(\mathbb{J})$, $b > 0$, and $\alpha \in [0,1)$. Applying Schaefer's theorem show that there exists a solution to the following nonlinear first order periodic problem

$$u'(t) + \mu u(t) = f(t,u(t)), \quad t \in \mathbb{J}, \quad u(0) = u(T).$$